DISCRETE RANDOM VARIABLES

DEFINITION : A *discrete random variable* is a *function* $X(s)$ from a *finite* or *countably infinite* sample space $\mathcal{S}$ to the real numbers :

$$X(\cdot) \quad : \quad \mathcal{S} \quad \rightarrow \quad \mathbb{R} \ .$$

EXAMPLE : Toss a coin 3 times in sequence. The sample space is

$$\mathcal{S} \ = \ \{HHH\ ,\ HHT\ ,\ HTH\ ,\ HTT\ ,\ THH\ ,\ THT\ ,\ TTH\ ,\ TTT\}\ ,$$

and examples of random variables are

- $X(s) \ = \$ the number of Heads in the sequence ; *e.g.*, $X(HTH) \ = \ 2$,

- $Y(s) \ = \$ The index of the first H ; *e.g.*, $Y(TTH) \ = \ 3$,

  0 if the sequence has no H , *i.e.*, $Y(TTT) \ = \ 0$.

NOTE : In this example $X(s)$ and $Y(s)$ are actually *integers* .

Value-ranges of a random variable correspond to *events* in $\mathcal{S}$.

EXAMPLE : For the sample space

$\mathcal{S} \ = \ \{HHH\ , HHT\ , HTH\ , HTT\ , THH\ , THT\ , TTH\ , TTT\}$,

with

$$X(s) \ = \ \text{the number of Heads} ,$$

the value

$X(s) \ = \ 2$, corresponds to the event $\{HHT\ ,\ HTH\ ,\ THH\}$,

and the values

$1 < X(s) \leq 3$, correspond to $\{HHH\ ,\ HHT\ ,\ HTH\ ,\ THH\}$.

NOTATION : If it is clear what $\mathcal{S}$ is then we often just write

$$X \qquad \text{instead of} \qquad X(s) .$$

Value-ranges of a random variable correspond to *events* in $\mathcal{S}$,

and

$$\textit{events} \text{ in } \mathcal{S} \text{ have a } \textit{probability} \ .$$

Thus

$$\textit{Value-ranges} \text{ of a random variable have a } \textit{probability} \ .$$

EXAMPLE : For the sample space

$$\mathcal{S} \ = \ \{HHH \, , \, HHT \, , \, HTH \, , \, HTT \, , \, THH \, , \, THT \, , \, TTH \, , \, TTT\} \, ,$$

with

$$X(s) \ = \ \text{the number of Heads} \ ,$$

we have

$$P(0 < X \le 2) \ = \ \frac{6}{8} \ .$$

QUESTION : What are the values of

$P(X \le -1)$, $P(X \le 0)$, $P(X \le 1)$, $P(X \le 2)$, $P(X \le 3)$, $P(X \le 4)$?

NOTATION : We will also write $p_X(x)$ to denote $P(X = x)$.

EXAMPLE : For the sample space

$$\mathcal{S} \;=\; \{HHH\,,\, HHT\,,\, HTH\,,\, HTT\,,\, THH\,,\, THT\,,\, TTH\,,\, TTT\}\,,$$

with

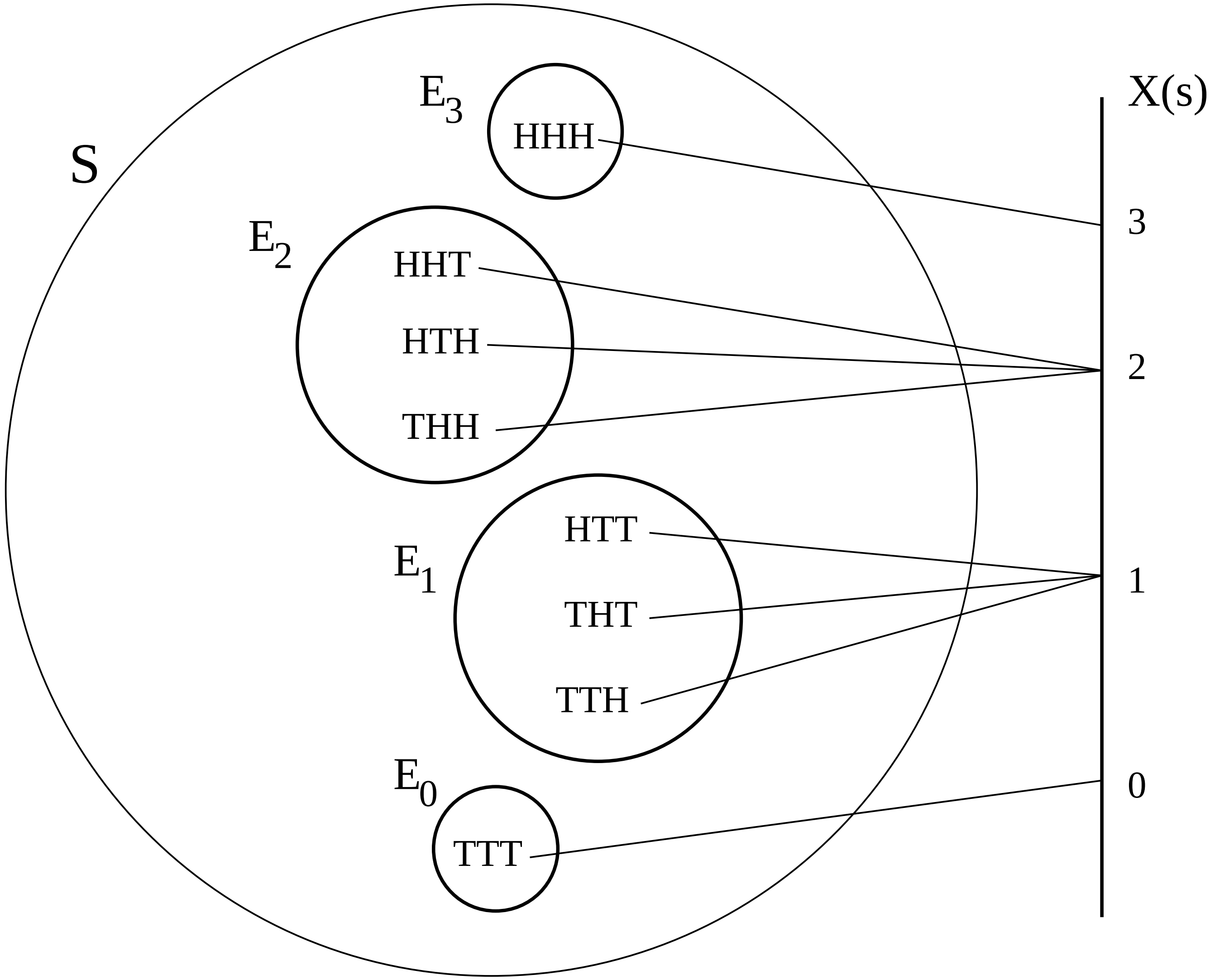
$$X(s) \;=\; \text{the number of Heads}\,,$$

we have

$$\begin{aligned}
p_X(0) &\equiv P(\ \{TTT\}\ ) &&= \tfrac{1}{8} \\
p_X(1) &\equiv P(\ \{HTT\,,\ THT\,,\ TTH\}\ ) &&= \tfrac{3}{8} \\
p_X(2) &\equiv P(\ \{HHT\,,\ HTH\,,\ THH\}\ ) &&= \tfrac{3}{8} \\
p_X(3) &\equiv P(\ \{HHH\}\ ) &&= \tfrac{1}{8}
\end{aligned}$$

where

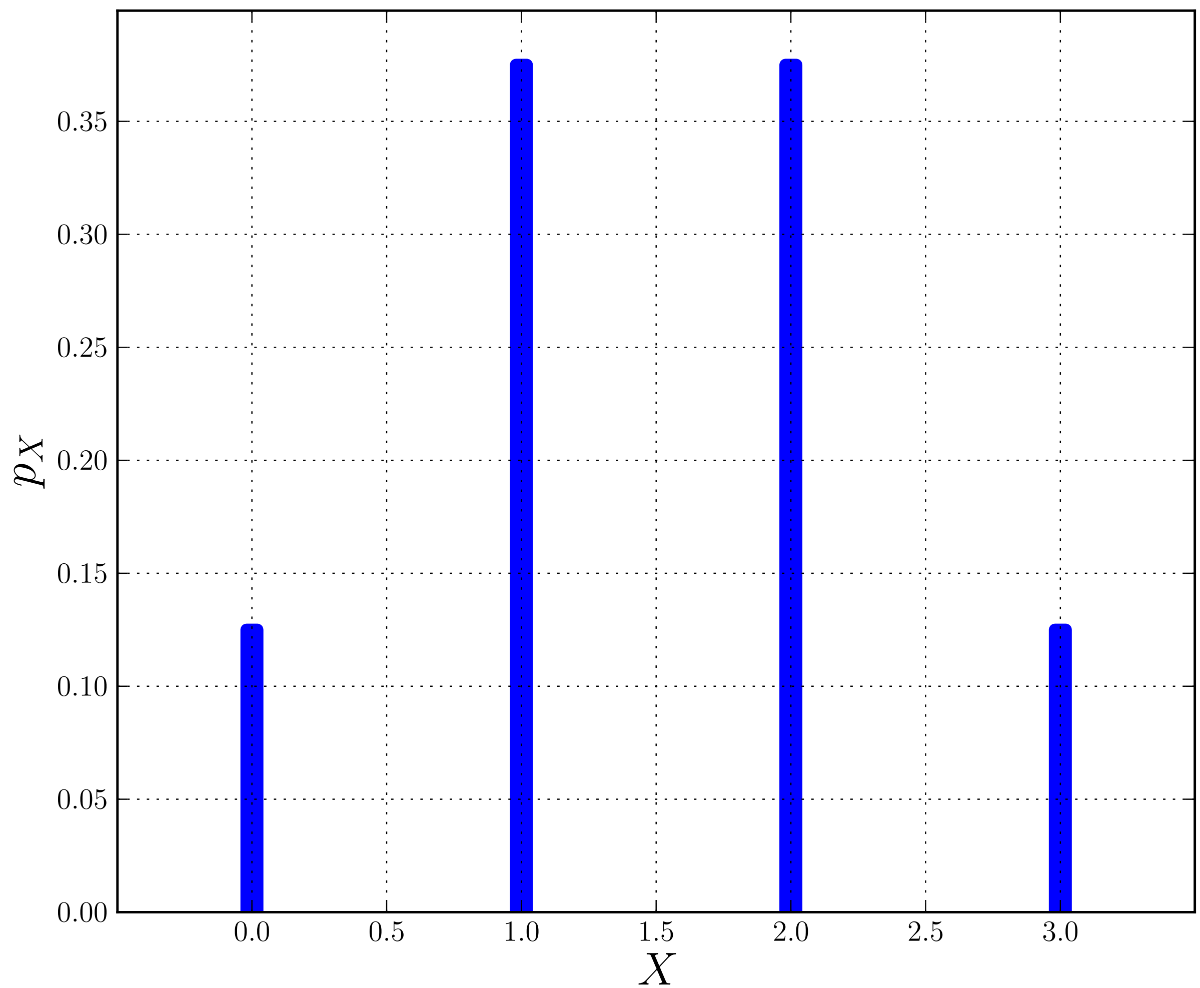
$$p_X(0) \;+\; p_X(1) \;+\; p_X(2) \;+\; p_X(3) \;=\; 1\,. \qquad (\text{ Why ? })$$



Graphical representation of X .

The *events* E_0, E_1, E_2, E_3 are *disjoint* since $X(s)$ is a *function* !

($X : S \rightarrow \mathbb{R}$ must be defined for *all* $s \in S$ and must be *single-valued*.)



The graph of p_X .

DEFINITION :

$$p_X(x) \equiv P(X = x) \ ,$$

is called the *probability mass function* .

DEFINITION :

$$F_X(x) \equiv P(X \leq x) \ ,$$

is called the (*cumulative*) *probability distribution function* .

PROPERTIES :

- $F_X(x)$ is a *non-decreasing* function of x . (Why ?)
- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. (Why ?)
- $P(a < X \leq b) = F_X(b) - F_X(a)$. (Why ?)

NOTATION : When it is clear what X is then we also write

$$p(x) \ \text{ for } \ p_X(x) \qquad \text{and} \qquad F(x) \ \text{ for } \ F_X(x) \ .$$

EXAMPLE : With $X(s)$ = the number of Heads , and

$$\mathcal{S} = \{HHH\,,\,HHT\,,\,HTH\,,\,HTT\,,\,THH\,,\,THT\,,\,TTH\,,\,TTT\}\,,$$

$$p(0) = \tfrac{1}{8} \quad , \quad p(1) = \tfrac{3}{8} \quad , \quad p(2) = \tfrac{3}{8} \quad , \quad p(3) = \tfrac{1}{8}\,,$$

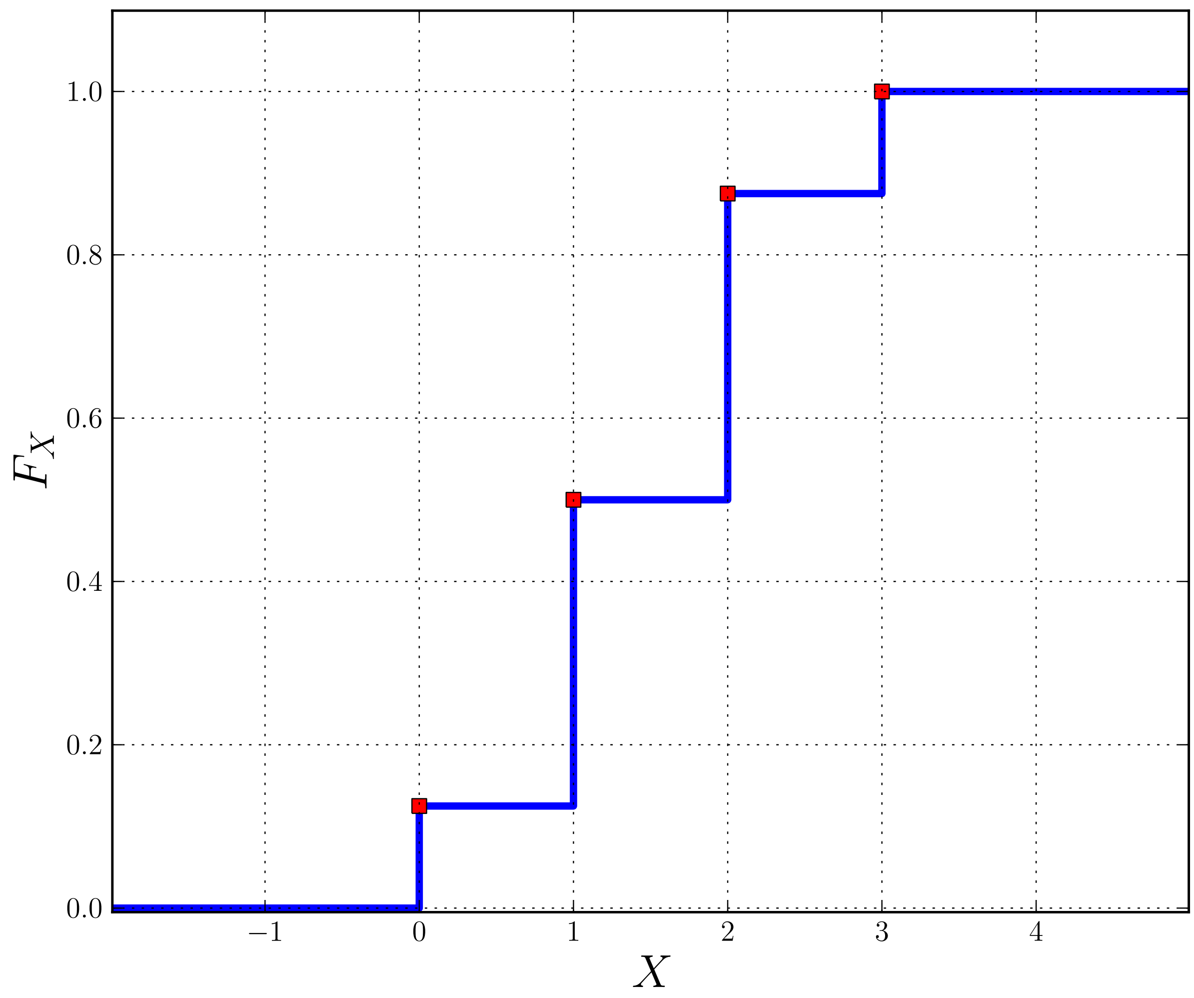
we have the *probability distribution function*

$$\begin{array}{lclcl} F(-1) & \equiv & P(X \leq -1) & = & 0 \\ F(\ 0) & \equiv & P(X \leq \ 0) & = & \mathbf{\tfrac{1}{8}} \\ F(\ 1) & \equiv & P(X \leq \ 1) & = & \tfrac{4}{8} \\ F(\ 2) & \equiv & P(X \leq \ 2) & = & \mathbf{\tfrac{7}{8}} \\ F(\ 3) & \equiv & P(X \leq \ 3) & = & 1 \\ F(\ 4) & \equiv & P(X \leq \ 4) & = & 1 \end{array}$$

We see, for example, that

$$P(0 < X \leq 2) \;=\; P(X = 1) \;+\; P(X = 2)$$

$$= \; F(2) \;-\; F(0) \;=\; \tfrac{7}{8} - \tfrac{1}{8} \;=\; \tfrac{6}{8}\,.$$



The graph of the *probability distribution function* F_X .

EXAMPLE : Toss a coin until "Heads" occurs.

Then the sample space is *countably infinite* , namely,

$$\mathcal{S} \;=\; \{H \;,\; TH \;,\; TTH \;,\; TTTH \;,\; \cdots \} \,.$$

The *random variable* X is the *number of rolls* until "Heads" occurs :

$$X(H) \;=\; 1 \quad , \quad X(TH) \;=\; 2 \quad , \quad X(TTH) \;=\; 3 \quad , \; \cdots$$

Then

$$p(1) = \tfrac{1}{2} \quad , \quad p(2) = \tfrac{1}{4} \quad , \quad p(3) = \tfrac{1}{8} \quad , \quad \cdots \qquad (\text{ Why ? })$$

and

$$F(n) \;=\; P(X \leq n) \;=\; \sum_{k=1}^{n} p(k) \;=\; \sum_{k=1}^{n} \frac{1}{2^k} \;=\; 1 - \frac{1}{2^n} \,,$$

and, as should be the case,

$$\sum_{k=1}^{\infty} p(k) \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} p(k) \;=\; \lim_{n\to\infty} \, (1 - \frac{1}{2^n}) \;=\; 1 \,.$$

NOTE : The outcomes in $\mathcal{S}$ *do not have equal probability* !

EXERCISE : Draw the *probability mass* and *distribution functions*.

$X(s)$ is the *number of tosses* until "Heads" occurs $\cdots$

REMARK : We can also take $\mathcal{S} \equiv \mathcal{S}_n$ as *all ordered outcomes of length* n. For example, for $n = 4$,

$$\mathcal{S}_4 = \{ \tilde{H}HHH , \quad \tilde{H}HHT , \quad \tilde{H}HTH , \quad \tilde{H}HTT ,$$

$$\tilde{H}THH , \quad \tilde{H}THT , \quad \tilde{H}TTH , \quad \tilde{H}TTT ,$$

$$T\tilde{H}HH , \quad T\tilde{H}HT , \quad T\tilde{H}TH , \quad T\tilde{H}TT ,$$

$$TT\tilde{H}H , \quad TT\tilde{H}T , \quad TTT\tilde{H} , \quad TTTT \quad \} .$$

where for each outcome the first "Heads" is marked as $\tilde{H}$.

Each outcome in $\mathcal{S}_4$ has *equal probability* 2^{-n} (here $2^{-4} = \frac{1}{16}$) , and

$$p_X(1) = \frac{1}{2} \quad , \quad p_X(2) = \frac{1}{4} \quad , \quad p_X(3) = \frac{1}{8} \quad , \quad p_X(4) = \frac{1}{16} \quad \cdots ,$$

independent of n .

Joint distributions

The *probability mass function* and the *probability distribution function* can also be functions of *more than one variable*.

EXAMPLE : Toss a coin 3 times in sequence. For the sample space

$$\mathcal{S} = \{HHH\,,\ HHT\,,\ HTH\,,\ HTT\,,\ THH\,,\ THT\,,\ TTH\,,\ TTT\}\,,$$

we let

$$X(s) = \#\ \text{Heads} \quad , \quad Y(s) = \text{index of the first } H \quad (0 \text{ for } TTT)\,.$$

Then we have the *joint probability mass function*

$$p_{X,Y}(x,y) = P(X = x\,,\ Y = y)\,.$$

For example,

$$\begin{aligned} p_{X,Y}(2,1) &= P(X = 2\,,\ Y = 1) \\ &= P(\text{ 2 Heads } ,\ 1^{\text{st}} \text{ toss is Heads}) \\ &= \tfrac{2}{8} = \tfrac{1}{4}\,. \end{aligned}$$

EXAMPLE : (continued $\cdots$) For

$\mathcal{S} = \{HHH\,, HHT\,, HTH\,, HTT\,, THH\,, THT\,, TTH\,, TTT\}\,,$

$X(s)$ = number of Heads, and $Y(s)$ = index of the first H ,

we can list the values of $p_{X,Y}(x,y)$:

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$\mathbf{y=1}$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathbf{x=2}$	0	$\mathbf{\frac{2}{8}}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

NOTE :

- The *marginal probability* p_X is the probability mass function of X.
- The *marginal probability* p_Y is the probability mass function of Y.

EXAMPLE : (continued $\cdots$)

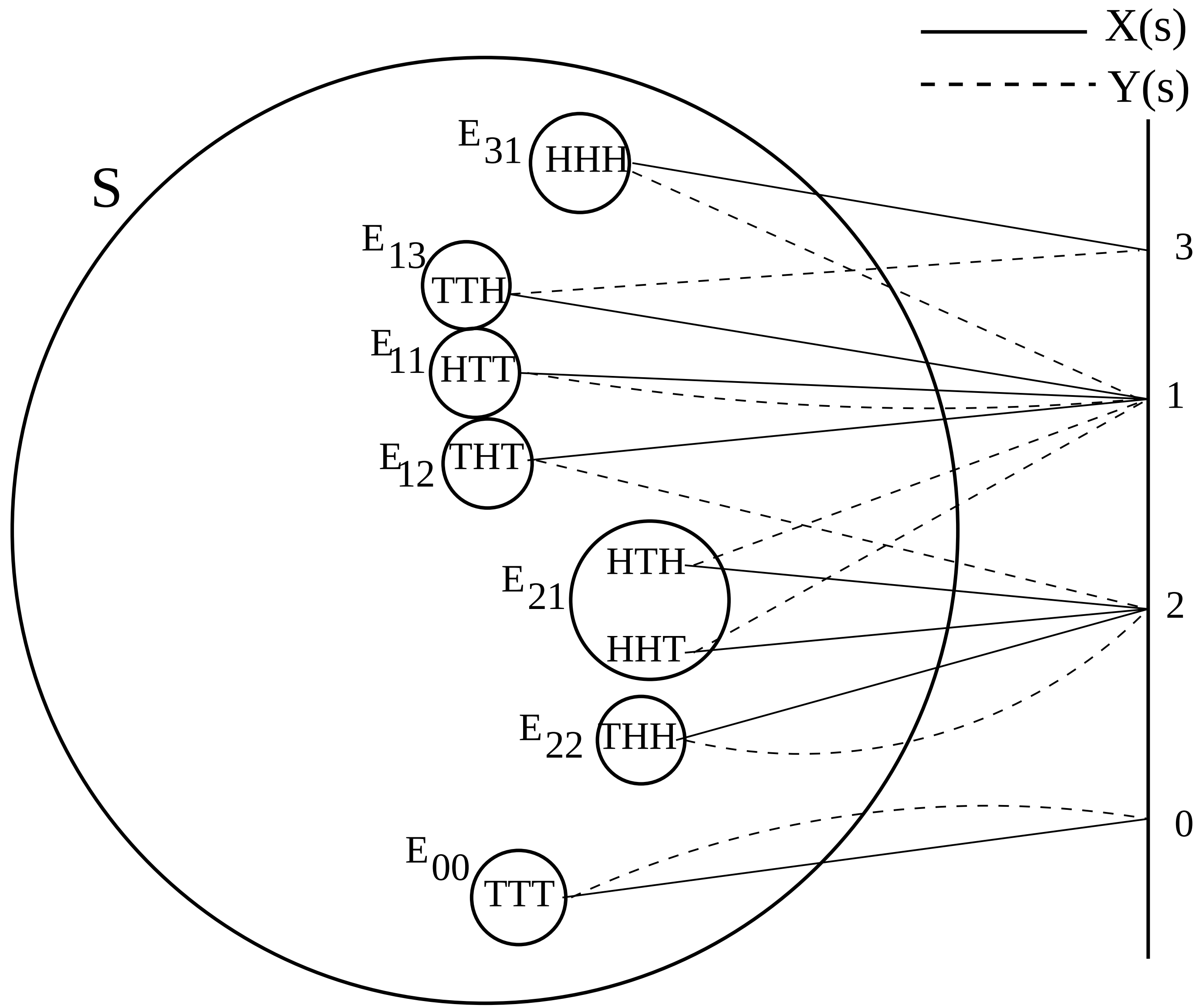
$X(s)$ = number of Heads, and $Y(s)$ = index of the first H .

	$y=0$	$\mathbf{y=1}$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathbf{x=2}$	0	$\mathbf{\frac{2}{8}}$	$\frac{1}{8}$	0	$\mathbf{\frac{3}{8}}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\mathbf{\frac{4}{8}}$	$\frac{2}{8}$	$\frac{1}{8}$	1

For example,

- $X = 2$ corresponds to the *event* $\{HHT\ ,\ HTH\ ,\ THH\}$.
- $Y = 1$ corresponds to the *event* $\{HHH\ ,\ HHT\ ,\ HTH\ ,\ HTT\}$.
- $(X = 2$ *and* $Y = 1)$ corresponds to the *event* $\{HHT\ ,\ HTH\}$.

QUESTION : Are the events $X = 2$ and $Y = 1$ *independent* ?



The *events* $E_{i,j} \equiv \{ s \in S : X(s) = i , Y(s) = j \}$ are *disjoint* .

QUESTION : Are the events $X = 2$ and $Y = 1$ *independent* ?

DEFINITION :

$$p_{X,Y}(x,y) \quad \equiv \quad P(X = x \ , \ Y = y) \ ,$$

is called the *joint probability mass function* .

DEFINITION :

$$F_{X,Y}(x,y) \quad \equiv \quad P(X \leq x \ , \ Y \leq y) \ ,$$

is called the *joint* (*cumulative*) *probability distribution function* .

NOTATION : When it is clear what X and Y are then we also write

$$p(x,y) \quad \text{for} \quad p_{X,Y}(x,y) \ ,$$

and

$$F(x,y) \quad \text{for} \quad F_{X,Y}(x,y) \ .$$

EXAMPLE : *Three* tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H .

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Joint distribution function $F_{X,Y}(x,y) \equiv P(X \leq x, Y \leq y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$F_X(\cdot)$
$x=0$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$x=1$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{4}{8}$
$x=2$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$x=3$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1
$F_Y(\cdot)$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1

Note that the distribution function F_X is a *copy* of the 4th column, and the distribution function F_Y is a *copy* of the 4th row. (Why ?)

In the preceding example :

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\mathbf{\frac{1}{8}}$	$\mathbf{0}$	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	$\mathbf{0}$	$\mathbf{0}$	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Joint distribution function $F_{X,Y}(x,y) \equiv P(X \leq x, Y \leq y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$F_X(\cdot)$
$x=0$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$x=1$	$\frac{1}{8}$	$\mathbf{\frac{2}{8}}$	$\frac{3}{8}$	$\mathbf{\frac{4}{8}}$	$\frac{4}{8}$
$x=2$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$x=3$	$\frac{1}{8}$	$\mathbf{\frac{5}{8}}$	$\frac{7}{8}$	$\mathbf{1}$	1
$F_Y(\cdot)$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	1	1

QUESTION : Why is

$$P(1 < X \leq 3\ , 1 < Y \leq 3) \ = \ F(3,3) - F(1,3) - F(3,1) + F(1,1)\ ?$$

EXERCISE :

Roll a *four-sided die* (tetrahedron) *two* times.

(The sides are marked 1 , 2 , 3 , 4 .)

Suppose each of the four sides is equally likely to end facing down.

Suppose the *outcome* of a *single roll* is the side that faces *down* (!).

Define the random variables X and Y as

$$X = \text{result of the } \textit{first roll} \quad , \quad Y = \textit{sum} \text{ of the two rolls.}$$

- What is a good choice of the *sample space* $\mathcal{S}$?
- How many outcomes are there in $\mathcal{S}$?
- List the values of the *joint probability mass function* $p_{X,Y}(x, y)$.
- List the values of the *joint cumulative distribution function* $F_{X,Y}(x, y)$.

EXERCISE :

Three balls are selected at random from a bag containing

$$2 \ \textit{red} \ , \ 3 \ \textit{green} \ , \ 4 \ \textit{blue} \ \text{balls} \ .$$

Define the *random variables*

$$R(s) \ = \ \text{the number of } \textit{red} \text{ balls drawn},$$

and

$$G(s) \ = \ \text{the number of } \textit{green} \text{ balls drawn} \ .$$

List the values of

- the *joint probability mass function* $p_{R,G}(r, g)$.
- the *marginal probability mass functions* $p_R(r)$ and $p_G(g)$.
- the *joint distribution function* $F_{R,G}(r, g)$.
- the *marginal distribution functions* $F_R(r)$ and $F_G(g)$.

Independent random variables

Two discrete random variables $X(s)$ and $Y(s)$ are *independent* if

$$P(X = x \ , \ Y = y) \ = \ P(X = x) \cdot P(Y = y) \ , \quad \text{for } \textit{all } x \text{ and } y \ ,$$

or, equivalently, if their *probability mass functions* satisfy

$$p_{X,Y}(x, y) \quad = \quad p_X(x) \ \cdot \ p_Y(y) \ , \quad \text{for } \textit{all } x \text{ and } y \ ,$$

or, equivalently, if the *events*

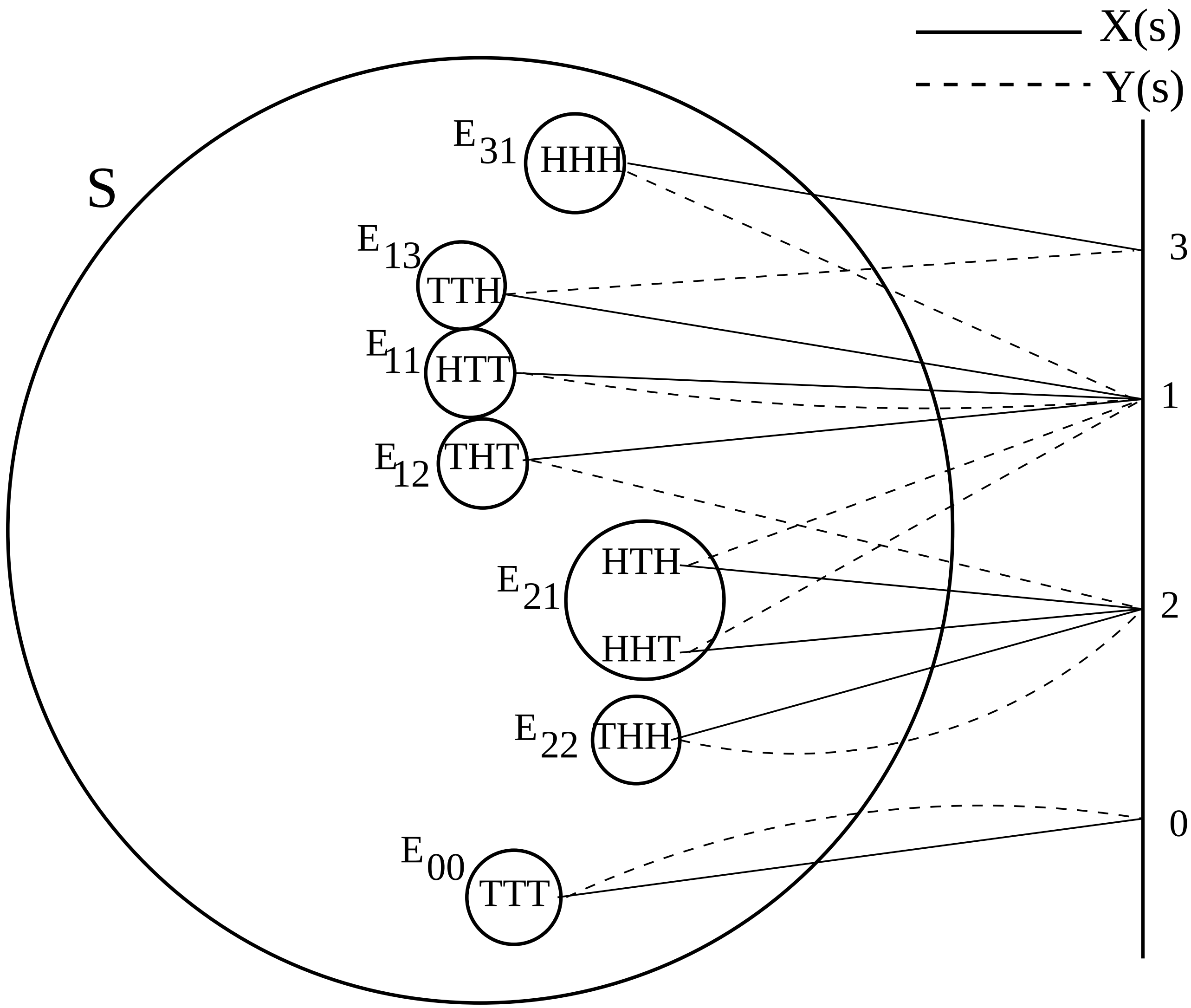
$$E_x \ \equiv \ X^{-1}(\{x\}) \quad \text{and} \quad E_y \ \equiv \ Y^{-1}(\{y\}) \ ,$$

are independent *in the sample space* $\mathcal{S}$, *i.e.*,

$$P(E_x E_y) \quad = \quad P(E_x) \cdot P(E_y) \ , \quad \text{for } \textit{all } x \text{ and } y \ .$$

NOTE :

- In the current *discrete* case, x and y are typically *integers* .
- $X^{-1}(\{x\}) \ \equiv \ \{ \ s \in \mathcal{S} \ : \ X(s) = x \ \}$.



Three tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H .

- What are the values of $p_X(2)$, $p_Y(1)$, $p_{X,Y}(2,1)$?
- Are X and Y *independent* ?

RECALL :

$X(s)$ and $Y(s)$ are *independent* if for all x and y :

$$p_{X,Y}(x,y) \quad = \quad p_X(x) \;\cdot\; p_Y(y) \; .$$

EXERCISE :

Roll a die two times in a row.

Let

$$X \text{ be the result of the } 1^{\text{st}} \text{ roll ,}$$

and

$$Y \text{ the result of the } 2^{\text{nd}} \text{ roll .}$$

Are X and Y *independent* , *i.e.*, is

$$p_{X,Y}(k,\ell) \;=\; p_X(k) \cdot p_Y(\ell), \qquad \text{for all} \;\; 1 \;\leq\; k,\ell \;\leq\; 6 \;\; ?$$

EXERCISE :

Are these random variables X and Y *independent* ?

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

EXERCISE : Are these random variables X and Y *independent* ?

Joint probability mass function $p_{X,Y}(x,y)$

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Joint distribution function $F_{X,Y}(x,y) \equiv P(X \leq x, Y \leq y)$

	$y=1$	$y=2$	$y=3$	$F_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{2}$
$x=2$	$\frac{5}{9}$	$\frac{25}{36}$	$\frac{5}{6}$	$\frac{5}{6}$
$x=3$	$\frac{2}{3}$	$\frac{5}{6}$	1	1
$F_Y(y)$	$\frac{2}{3}$	$\frac{5}{6}$	1	1

QUESTION : Is $F_{X,Y}(x,y) \;=\; F_X(x) \cdot F_Y(y)$?

PROPERTY :

The *joint distribution function* of *independent* random variables X and Y satisfies

$$F_{X,Y}(x,y) \quad = \quad F_X(x) \cdot F_Y(y) \ , \qquad \text{for all } x, y \ .$$

PROOF :

$$\begin{aligned}
F_{X,Y}(x_k,y_\ell) \quad &= \quad P(X \leq x_k \ , \ Y \leq y_\ell) \\
&= \quad \textstyle\sum_{i \leq k} \ \sum_{j \leq \ell} \ p_{X,Y}(x_i, y_j) \\
&= \quad \textstyle\sum_{i \leq k} \ \sum_{j \leq \ell} \ p_X(x_i) \cdot p_Y(y_j) \qquad \text{(by independence)} \\
&= \quad \textstyle\sum_{i \leq k} \ \{ \ p_X(x_i) \cdot \sum_{j \leq \ell} \ p_Y(y_j) \ \} \\
&= \quad \{ \textstyle\sum_{i \leq k} \ p_X(x_i) \ \} \ \cdot \ \{ \sum_{j \leq \ell} \ p_Y(y_j) \ \} \\
&= \quad F_X(x_k) \cdot F_Y(y_\ell) \ .
\end{aligned}$$

Conditional distributions

Let X and Y be discrete random variables with *joint probability mass function*

$$p_{X,Y}(x,y) \ .$$

For given x and y , let

$$E_x \ = \ X^{-1}(\{x\}) \qquad \text{and} \qquad E_y \ = \ Y^{-1}(\{y\}) \ ,$$

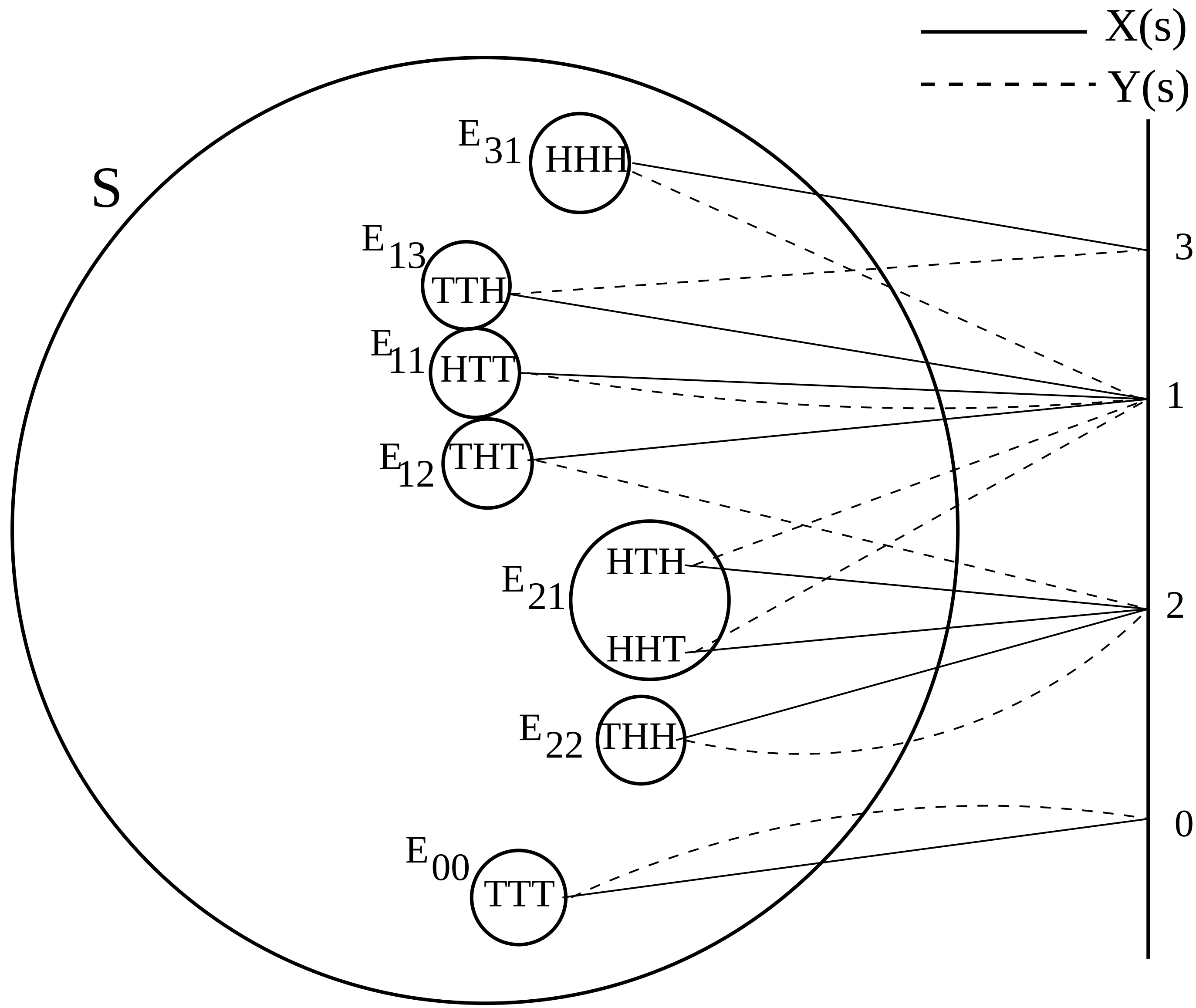
be their corresponding *events* in the sample space $\mathcal{S}$.

Then

$$P(E_x|E_y) \ \equiv \ \frac{P(E_xE_y)}{P(E_y)} \ = \ \frac{p_{X,Y}(x,y)}{p_Y(y)} \ .$$

Thus it is natural to define the *conditional probability mass function*

$$p_{X|Y}(x|y) \ \equiv \ P(X=x \mid Y=y) \ = \ \frac{p_{X,Y}(x,y)}{p_Y(y)} \ .$$



Three tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H .

- What are the values of $P(X = 2 \mid Y = 1)$ and $P(Y = 1 \mid X = 2)$?

EXAMPLE : (3 tosses : $X(s) = \#$ Heads, $Y(s) =$ index 1^{st} H.)

Joint probability mass function $p_{X,Y}(x,y)$

	$y=0$	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x=1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x=2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x=3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

Conditional probability mass function $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$.

	$y=0$	$y=1$	$y=2$	$y=3$
$x=0$	1	0	0	0
$x=1$	0	$\frac{2}{8}$	$\frac{4}{8}$	1
$x=2$	0	$\frac{4}{8}$	$\frac{4}{8}$	0
$x=3$	0	$\frac{2}{8}$	0	0
	1	1	1	1

EXERCISE : Also construct the Table for $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$.

EXAMPLE :

Joint probability mass function $p_{X,Y}(x, y)$

	$y = 1$	$y = 2$	$y = 3$	$p_X(x)$
$x = 1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x = 2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x = 3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Conditional probability mass function $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$.

	$y = 1$	$y = 2$	$y = 3$
$x = 1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$x = 2$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$x = 3$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	1	1	1

QUESTION : What does the last Table tell us?

EXERCISE : Also construct the Table for $P(Y = y|X = x)$.

Expectation

The *expected value* of a discrete random variable X is

$$E[X] \quad \equiv \quad \sum_k x_k \cdot P(X = x_k) \quad = \quad \sum_k x_k \cdot p_X(x_k) \ .$$

Thus $E[X]$ represents the *weighted average value* of X .

($E[X]$ is also called the *mean* of X .)

EXAMPLE : The *expected value* of *rolling a die* is

$$E[X] \quad = \quad 1 \cdot \tfrac{1}{6} \ + \ 2 \cdot \tfrac{1}{6} \ + \ \cdots \ + \ 6 \cdot \tfrac{1}{6} \quad = \quad \tfrac{1}{6} \cdot \textstyle\sum_{k=1}^{6} k \quad = \quad \tfrac{7}{2} \ .$$

EXERCISE : Prove the following :

- $E[aX] \quad = \quad a \ E[X] \ ,$
- $E[aX + b] \quad = \quad a \ E[X] \ + \ b \ .$

EXAMPLE : Toss a coin until "Heads" occurs. Then

$$\mathcal{S} = \{ H \ , \ TH \ , \ TTH \ , \ TTTH \ , \ \cdots \} \ .$$

The *random variable* X is the *number of tosses* until "Heads" occurs :

$$X(H) = 1 \ , \quad X(TH) = 2 \ , \quad X(TTH) = 3 \ .$$

Then

$$E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2^k} = 2 \ .$$

n	$\sum_{k=1}^{n} k/2^k$
1	0.50000000
2	1.00000000
3	1.37500000
10	1.98828125
40	1.99999999

REMARK :

Perhaps using $\mathcal{S}_n = \{$all sequences of n tosses$\}$ is better $\cdots$

The expected value of a *function of a random variable* is

$$E[g(X)] \equiv \sum_k g(x_k)\, p(x_k) .$$

EXAMPLE :

The *pay-off* of rolling a die is $\$k^2$, where k is the side facing up.

What should the *entry fee* be for the betting to break even?

SOLUTION : Here $g(X) = X^2$, and

$$E[g(X)] = \sum_{k=1}^{6} k^2 \frac{1}{6} = \frac{1}{6} \frac{6(6+1)(2 \cdot 6 + 1)}{6} = \frac{91}{6} \cong \$15.17 .$$

The expected value of a function of *two* random variables is

$$E[g(X,Y)] \equiv \sum_k \sum_\ell g(x_k, y_\ell)\ p(x_k, y_\ell)\ .$$

EXAMPLE :

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

$$E[X] = 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{3} + 3 \cdot \tfrac{1}{6} = \mathbf{\tfrac{5}{3}}\ ,$$

$$E[Y] = 1 \cdot \tfrac{2}{3} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} = \mathbf{\tfrac{3}{2}}\ ,$$

$$\begin{aligned} E[XY] = {} & 1 \cdot \tfrac{1}{3} + 2 \cdot \tfrac{1}{12} + 3 \cdot \tfrac{1}{12} \\ & + 2 \cdot \tfrac{2}{9} + 4 \cdot \tfrac{1}{18} + 6 \cdot \tfrac{1}{18} \\ & + 3 \cdot \tfrac{1}{9} + 6 \cdot \tfrac{1}{36} + 9 \cdot \tfrac{1}{36} = \mathbf{\tfrac{5}{2}}\ . \end{aligned}$$

(So ?)

PROPERTY :

- If X and Y are *independent* then $E[XY] = E[X]\ E[Y]$.

PROOF :

$$
\begin{aligned}
E[XY] &= \textstyle\sum_k \sum_\ell \ x_k \ y_\ell \ p_{X,Y}(x_k, y_\ell) \\
&= \textstyle\sum_k \sum_\ell \ x_k \ y_\ell \ p_X(x_k) \ p_Y(y_\ell) \qquad \text{(by independence)} \\
&= \textstyle\sum_k \{ \ x_k \ p_X(x_k) \sum_\ell \ y_\ell \ p_Y(y_\ell) \} \\
&= \{ \textstyle\sum_k \ x_k \ p_X(x_k) \} \cdot \{ \sum_\ell \ y_\ell \ p_Y(y_\ell) \} \\
&= E[X] \cdot E[Y] \ .
\end{aligned}
$$

EXAMPLE : See the preceding example !

PROPERTY : $E[X+Y] \; = \; E[X] \; + \; E[Y]$. (**Always** !)

PROOF :

$$
\begin{aligned}
E[X+Y] &= \textstyle\sum_k \sum_\ell \; (x_k + y_\ell) \; p_{X,Y}(x_k, y_\ell) \\
&= \textstyle\sum_k \sum_\ell \; x_k \; p_{X,Y}(x_k, y_\ell) \; + \; \sum_k \sum_\ell \; y_\ell \; p_{X,Y}(x_k, y_\ell) \\
&= \textstyle\sum_k \sum_\ell \; x_k \; p_{X,Y}(x_k, y_\ell) \; + \; \sum_\ell \sum_k \; y_\ell \; p_{X,Y}(x_k, y_\ell) \\
&= \textstyle\sum_k \{ x_k \sum_\ell \; p_{X,Y}(x_k, y_\ell) \} \; + \; \sum_\ell \{ \; y_\ell \sum_k \; p_{X,Y}(x_k, y_\ell) \} \\
&= \textstyle\sum_k \{ x_k \; p_X(x_k) \} \; + \; \sum_\ell \{ y_\ell \; p_Y(y_\ell) \} \\
&= E[X] \; + \; E[Y] \; .
\end{aligned}
$$

NOTE : X and Y need not be independent !

EXERCISE :

Probability mass function $p_{X,Y}(x,y)$

	$y=6$	$y=8$	$y=10$	$p_X(x)$
$x=1$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$x=2$	0	$\frac{1}{5}$	0	$\frac{1}{5}$
$x=3$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$p_Y(y)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	1

Show that

- $E[X] = 2$, $E[Y] = 8$, $E[XY] = 16$
- X and Y are *not* independent

Thus if

$$E[XY] = E[X]\,E[Y]\,,$$

then it does not necessarily follow that X and Y are independent !

Variance and Standard Deviation

Let X have *mean*

$$\mu \;=\; E[X] \; .$$

Then the *variance* of X is

$$Var(X) \;\equiv\; E[\,(X-\mu)^2] \;\equiv\; \sum_k \,(x_k-\mu)^2 \; p(x_k) \; ,$$

which is the average weighted *square distance* from the mean.

We have

$$\begin{aligned} Var(X) \quad &= \quad E[X^2 \;-\; 2\mu X \;+\; \mu^2] \\ &= \quad E[X^2] \;-\; 2\mu E[X] \;+\; \mu^2 \\ &= \quad E[X^2] \;-\; 2\mu^2 \;+\; \mu^2 \\ &= \quad E[X^2] \;-\; \mu^2 \; . \end{aligned}$$

The *standard deviation* of X is

$$\sigma(X) \equiv \sqrt{Var(X)} = \sqrt{E[\,(X-\mu)^2]} = \sqrt{E[X^2] - \mu^2} \,.$$

which is the average weighted *distance* from the mean.

EXAMPLE : The *variance* of *rolling a die* is

$$Var(X) = \sum_{k=1}^{6} [k^2 \cdot \frac{1}{6}] - \mu^2$$

$$= \frac{1}{6} \frac{6(6+1)(2\cdot 6+1)}{6} - (\frac{7}{2})^2 = \frac{35}{12} \,.$$

The *standard deviation* is

$$\sigma = \sqrt{\frac{35}{12}} \cong 1.70 \,.$$

Covariance

Let X and Y be random variables with *mean*

$$E[X] = \mu_X \quad , \quad E[Y] = \mu_Y \ .$$

Then the *covariance* of X and Y is defined as

$$Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] = \sum_{k,\ell} (x_k-\mu_X)\,(y_\ell-\mu_Y)\,p(x_k,y_\ell)\ .$$

We have

$$\begin{aligned} Cov(X,Y) &= E[\,(X-\mu_X)\,(Y-\mu_Y)\,] \\ &= E[XY - \mu_X Y - \mu_Y X \ + \ \mu_X\mu_Y] \\ &= E[XY] \ - \ \mu_X\mu_Y \ - \ \mu_Y\mu_X \ + \ \mu_X\mu_Y \\ &= E[XY] \ - \ E[X]\ E[Y]\ . \end{aligned}$$

We defined

$$\begin{aligned} Cov(X,Y) &\equiv E[\ (X - \mu_X)\ (Y - \mu_Y)\ ] \\ &= \sum_{k,\ell} (x_k - \mu_X)\ (y_\ell - \mu_Y)\ p(x_k, y_\ell) \\ &= E[XY] \ - \ E[X]\ E[Y]\ . \end{aligned}$$

NOTE :

$Cov(X,Y)$ measures "*concordance*" or "*coherence*" of X and Y :

- If $X > \mu_X$ when $Y > \mu_Y$ and $X < \mu_X$ when $Y < \mu_Y$ then

$$Cov(X,Y) \ > \ 0\ .$$

- If $X > \mu_X$ when $Y < \mu_Y$ and $X < \mu_X$ when $Y > \mu_Y$ then

$$Cov(X,Y) \ < \ 0\ .$$

EXERCISE : Prove the following :

- $Var(aX + b) \quad = \quad a^2 \, Var(X)$,

- $Cov(X, Y) \quad = \quad Cov(Y, X)$,

- $Cov(cX, Y) \quad = \quad c \, Cov(X, Y)$,

- $Cov(X, cY) \quad = \quad c \, Cov(X, Y)$,

- $Cov(X + Y, Z) \quad = \quad Cov(X, Z) \; + \; Cov(Y, Z)$,

- $Var(X + Y) \quad = \quad Var(X) \; + \; Var(Y) \; + \; 2 \, Cov(X, Y)$.

PROPERTY :

If X and Y are *independent* then $Cov(X,Y) = 0$.

PROOF :

We have already shown (with $\mu_X \equiv E[X]$ and $\mu_Y \equiv E[Y]$) that

$$Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,] = E[XY] - E[X]\,E[Y] ,$$

and that if X and Y are *independent* then

$$E[XY] = E[X]\,E[Y] .$$

from which the result follows.

EXERCISE : (already used earlier $\cdots$)

Probability mass function $p_{X,Y}(x,y)$

	$y = 6$	$y = 8$	$y = 10$	$p_X(x)$
$x = 1$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$x = 2$	0	$\frac{1}{5}$	0	$\frac{1}{5}$
$x = 3$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$p_Y(y)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	1

Show that

- $E[X] = 2 \quad , \quad E[Y] = 8 \quad , \quad E[XY] = 16$
- $Cov(X,Y) = E[XY] - E[X]\,E[Y] = 0$
- X and Y are *not* independent

Thus if

$$Cov(X,Y) = 0 \,,$$

then it does not necessarily follow that X and Y are independent !

PROPERTY :

If X and Y are *independent* then

$$Var(X + Y) \;=\; Var(X) \;+\; Var(Y) \; .$$

PROOF :

We have already shown (in an exercise !) that

$$Var(X + Y) \;=\; Var(X) \;+\; Var(Y) \;+\; 2\, Cov(X, Y) \; ,$$

and that if X and Y are *independent* then

$$Cov(X, Y) \;=\; 0 \; ,$$

from which the result follows.

EXERCISE :

Compute

$$E[X] \ , \ E[Y] \ , \ E[X^2] \ , \ E[Y^2]$$

$$E[XY] \ , \ Var(X) \ , \ Var(Y)$$

$$Cov(X,Y)$$

for

Joint probability mass function $p_{X,Y}(x,y)$

	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$p_X(x)$
$x = 0$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
$x = 1$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$x = 2$	0	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{3}{8}$
$x = 3$	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$p_Y(y)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	1

EXERCISE :

Compute

$$E[X] \quad , \quad E[Y] \quad , \quad E[X^2] \quad , \quad E[Y^2]$$

$$E[XY] \quad , \quad Var(X) \quad , \quad Var(Y)$$

$$Cov(X,Y)$$

for

Joint probability mass function $p_{X,Y}(x,y)$

	$y=1$	$y=2$	$y=3$	$p_X(x)$
$x=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$x=2$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{3}$
$x=3$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1